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# Indirect sampled-data control with sampling period adaptation

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**Abstract.** It is known that if a continuous-time feedback system is exponentially stable, then the corresponding sampled-data system obtained by sample-and-hold discretization with constant sampling period is also exponentially stable, provided that the sampling period  $\tau > 0$  is sufficiently small. In general it is difficult to estimate how small the sampling period has to be in order to achieve stability of the sampled-data system. In this paper, we present an adaptive mechanism for adjusting the sampling period. This mechanism has the properties that, for every initial state, (i) the adaptation of the sampling period terminates after finitely many time steps and (ii) the state of the adaptive sampled-data system is integrable and converges to zero as time goes to infinity.

**Keywords.** Adaptive control, feedback stabilization, indirect sampled-data control, variable sampling period.

## 1 Introduction

Consider the finite-dimensional continuous-time static output feedback system

$$\left. \begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t); & x(0) &= x^0, \\ y(t) &= Cx(t), \\ u(t) &= Fy(t), \end{aligned} \right\} \quad (1.1)$$

where  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $C \in \mathbb{R}^{p \times n}$ ,  $F \in \mathbb{R}^{m \times p}$  and  $x^0 \in \mathbb{R}^n$ . System (1.1) is exponentially stable if, and only if, the matrix  $A + BFC$  is exponentially stable, that is, all eigenvalues of  $A + BFC$  have negative real parts.

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Digital implementation of the output feedback in (1.1) requires the application of sampling and (zero-order) hold, leading to the sampled-data feedback system

$$\left. \begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t); \quad x(0) = x^0, \\ y(t) &= Cx(t), \\ u(t) &= Fy(j\tau), \end{aligned} \right\} \quad \forall t \in [j\tau, (j+1)\tau), \quad (1.2)$$

where  $\tau > 0$  is the sampling period. It is well known that if system (1.1) is exponentially stable and if sampling period  $\tau$  is sufficiently small, then system (1.2) is also exponentially stable in the sense that there exist  $M \geq 1$  and  $\alpha > 0$  such that

$$\|x(t; x^0, \tau)\| \leq Me^{-\alpha t} \|x^0\|, \quad \forall x^0 \in \mathbb{R}^n \quad \forall t \geq 0,$$

where  $x(\cdot; x^0, \tau)$  denotes the solution of (1.2) (for the proof and for related results, see [1, 2, 5, 6]).

Given that the continuous-time system (1.1) is exponentially stable, it is in general difficult to estimate how small the sampling period has to be in order to achieve stability of the sampled-data system (1.2) (see [10]). In this paper, we develop an adaptive strategy for adjusting the sampling period, so that, for every initial condition  $x^0$ , the adaptation of the sampling period terminates after finitely many time steps and the corresponding solution of (1.2) is integrable and tends to 0 as  $t \rightarrow \infty$ .

The idea to invoke sampling period adaptation in the synthesis of stable sampled-data feedback systems seems to have been introduced in [7], where it is used in a high-gain control context. The approach in [7], developed for single-input, single-output minimum phase systems with relative degree one, was extended in [4] to include multi-input, multi-output systems. Additionally, a number of other assumptions imposed in [7] were relaxed in [4]. Furthermore, sampling period adaptation has also been used in [8] in a low-gain integral control context. However, the results in [4, 7] and in [8] are specific to high-gain stabilization and low-gain tracking, respectively, and have little overlap with the general result on adaptive sampling in indirect sampled-data control presented in the current paper.

The rest of the paper is structured as follows. Section 2 is devoted to the statement, discussion and illustration (by means of an example) of Theorem 2.2, the main result of the paper. Whilst Theorem 2.2 is restricted to static output feedback, it is shown in Section 3 how it can be extended to indirect sampled-data control involving dynamic feedback. All proofs can be found in Section 4. Finally, some conclusions are drawn in Section 5.

## Nomenclature and terminology.

$\lfloor \sigma \rfloor$	$:= \max\{n \in \mathbb{N}_0 \mid n \leq \sigma\}, \quad \sigma \in \mathbb{R}_+,$
$\ell^\infty(\mathbb{N}_0, \mathbb{R}^n)$	space of bounded $\mathbb{R}^n$ -valued sequences $(s_j)_{j \in \mathbb{N}_0}$ ,
$\ell^1(\mathbb{N}_0, \mathbb{R}^n)$	space of $\mathbb{R}^n$ -valued sequences $(s_j)_{j \in \mathbb{N}_0}$ with $\sum_{j=0}^\infty \ s_j\  < \infty$ ,
$L^1(\mathbb{R}_+, \mathbb{R}^n)$	vector space of all measurable functions $f : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ with $\int_0^\infty \ f(t)\  dt < \infty$ .

A sequence  $(s_j)_{j \in \mathbb{N}_0}$  is said to be *ultimately constant* if, and only if, there exists  $N \in \mathbb{N}_0$  such that  $s_{N+j} = s_N$  for all  $j \in \mathbb{N}_0$ .

## 2 Adaptation of the sampling period

The purpose of this section is to develop an adaptive feedback mechanism for adjusting the sampling period. The use of sampling and hold in (1.1), corresponding to the sampling points  $(t_j)_{j \in \mathbb{N}_0}$ , leads to the following sampled-data feedback system

$$\left. \begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t); \quad x(0) = x^0, \\ y(t) &= Cx(t), \\ u(t) &= Fy(t_j), \end{aligned} \right\} \quad \forall t \in [t_j, t_{j+1}). \quad (2.1)$$

The sampling points  $t_j$ , or, equivalently, the sampling periods  $\tau_j := t_{j+1} - t_j$ , are determined by the following adaptive strategy:

$$\left. \begin{aligned} \text{for given } & \alpha \in (0, 1) \text{ and } (\eta_j)_{j \in \mathbb{N}_0} \in \ell^\infty(\mathbb{N}_0, \mathbb{R}) \text{ with } \inf_{j \in \mathbb{N}_0} \eta_j > 0, \\ \text{set } & t_0 = 0, \quad \text{let } \sigma_0 \geq 0, \\ \text{and, for } & j = 0, 1, 2, \dots, \quad \text{set} \\ & k_j = \lfloor \sigma_j \rfloor, \\ & \tau_j = \max \{ \eta_j / (j+1)^\alpha, \eta_{k_j} / (k_j+1)^\alpha \}, \\ & t_{j+1} = t_j + \tau_j, \\ & \sigma_{j+1} = \sigma_j + \|y(t_j)\|. \end{aligned} \right\} \quad (2.2)$$

The rationale for the adaptive strategy (2.2) is described in the following remark.

**Remark 2.1. (i)** The choice of  $(\eta_j)_{j \in \mathbb{N}_0}$  and  $\alpha$  in (2.2) allows to influence the size of the sampling periods  $\tau_j$  in the transient phase where  $j$  is “small”: for example, the larger  $\eta_j$ , the larger  $\tau_j$  and similarly, the smaller  $\alpha$ , the larger  $\tau_j$ . Moreover, we emphasize that the sequence  $(\eta_j)_{j \in \mathbb{N}_0}$  plays a further role which will become clear later: see part (iii) of Remark 2.8.

**(ii)** Obviously, the last line in (2.2) (the recursion for  $\sigma_j$ ) is a discrete-time integrator with initial state  $\sigma_0$  and input  $(\|y(t_j)\|)_{j \in \mathbb{N}_0}$ , so that

$$\sigma_j = \sigma_0 + \sum_{l=0}^{j-1} \|y(t_l)\|, \quad \forall j \in \mathbb{N}. \quad (2.3)$$

It is immediate that the following properties are equivalent:

- (a) the sequence  $(\tau_j)_{j \in \mathbb{N}_0}$  is ultimately constant;
- (b) the sequence  $(k_j)_{j \in \mathbb{N}_0}$  is ultimately constant;
- (c)  $(\sigma_j)_{j \in \mathbb{N}_0} \in \ell^\infty(\mathbb{N}_0, \mathbb{R})$ ;
- (d)  $(y(t_j))_{j \in \mathbb{N}_0} \in \ell^1(\mathbb{N}_0, \mathbb{R}^p)$ .

We note that if  $(\tau_j)_{j \in \mathbb{N}_0}$  is not ultimately constant, then  $\lim_{j \rightarrow \infty} \tau_j = 0$  (as follows from the equivalence of (a) and (b)). Furthermore, we see that if the sequence  $(\eta_j)_{j \in \mathbb{N}_0}$  is non-increasing (a natural choice), then  $(\tau_j)_{j \in \mathbb{N}_0}$  is non-increasing. The idea behind (2.2) is to drive  $\tau_j$  to zero as long as the norm of the sampled output values  $y(t_j)$  is “large” in the sense that the partial sums  $\sigma_j$  has not “started to converge”.  $\diamond$

For the following, it is convenient to define

$$\delta_l := \eta_l / (l + 1)^\alpha, \quad \forall l \in \mathbb{N}_0. \quad (2.4)$$

Note that, for each sampling period  $\tau_j$  generated by (2.2), there exists  $l_j \in \mathbb{N}_0$  such that  $\tau_j = \delta_{l_j}$ . We introduce the following detectability hypothesis.

**(D)** The pair  $(C, e^{A\delta_l})$  is discrete-time detectable for every  $l \in \mathbb{N}_0$ .

We are now ready to state the main result of this contribution. The proof can be found in Section 4.

**Theorem 2.2.** *Assume that the continuous-time feedback system (1.1) is exponentially stable and let  $x(\cdot; x^0)$  denote the solution of the adaptive sampled-data system given by (2.1) and (2.2). Then, for every initial state  $x^0 \in \mathbb{R}^n$ , the following statements hold:*

- (i) *the sequence  $(\tau_j)_{j \in \mathbb{N}_0}$  is ultimately constant, that is, the adaptation of the sampling period terminates in finite time;*
- (ii) *if, additionally, hypothesis (D) is satisfied, then  $\lim_{t \rightarrow \infty} x(t; x^0) = 0$ ,  $x(\cdot; x^0) \in L^1(\mathbb{R}_+, \mathbb{R}^n)$  and  $(x(t_j; x^0))_{j \in \mathbb{N}_0} \in \ell^1(\mathbb{N}_0, \mathbb{R}^n)$ .*

Note that, by part (i) of Theorem 2.2, the limit  $\tau := \lim_{j \rightarrow \infty} \tau_j$  exists. Whilst Theorem 2.2 guarantees that, for every  $x^0 \in \mathbb{R}^n$ ,  $x(t; x^0) \rightarrow 0$  as  $t \rightarrow \infty$ , it does not ensure that the sampled-data feedback system (2.1) with constant sampling period  $\tau$  is asymptotically stable or, equivalently, that the spectral radius of the matrix

$$\Delta_\tau := e^{A\tau} + \int_0^\tau e^{As} ds BFC \quad (2.5)$$

is smaller than 1, as the following trivial example shows.

**Example 2.3.** Let  $C = I$  (in which case, for every sequence  $(\eta_j)_{j \in \mathbb{N}_0}$  and every  $\alpha$ , hypothesis (D) is trivially satisfied). Choose  $A$ ,  $B$ ,  $F$  and  $\tau > 0$  such that  $A + BF$  is Hurwitz and  $\Delta_\tau$  has at least one eigenvalue  $\lambda_u$  with  $|\lambda_u| \geq 1$  and at least one eigenvalue  $\lambda_s$  with  $|\lambda_s| < 1$ . Let  $v_s$  be in the  $\lambda_s$ -eigenspace of  $\Delta_\tau$  with

$$\|v_s\| < 1 - |\lambda_s|. \quad (2.6)$$

With  $\eta_j = \tau$  for all  $j \in \mathbb{N}_0$ ,  $\alpha \in (0, 1)$  arbitrary,  $x^0 = v_s$  and  $\sigma_0 = 0$ , it follows easily that the adaptive sampled-data system given by (2.1) and (2.2) has the following properties:  $\tau_j = \tau$  for all  $j \in \mathbb{N}_0$ ,  $y(t_j) = x(t_j; x^0) = x(j\tau; x^0) = \lambda_s^j v_s$  and  $k_j = 0$  for all  $j \in \mathbb{N}_0$ . This can be shown by an elementary induction argument combined with the observation that

$$\sigma_j = \sum_{l=0}^{j-1} |\lambda_s|^l \|v_s\| \leq \frac{1}{1 - |\lambda_s|} \|v_s\| < 1, \quad \forall j \in \mathbb{N},$$

which is a consequence of (2.3) and (2.6). ◇

The phenomenon described in Example 2.3 is reminiscent of the well-known fact that, in adaptive stabilization, the limiting feedback controller is not necessarily stabilizing, see [11, 12] and the references therein for more details.

**Remark 2.4.** As has already been indicated in the Introduction: given a feedback matrix  $F$  rendering the continuous-time system (1.1) exponentially stable, it is a difficult task to derive conditions (in terms of  $A$ ,  $B$ ,  $C$  and  $F$ ) for a sampling period  $\tau^*$  guaranteeing that the sampled-data system (1.2) is asymptotically stable for every (fixed) sampling period  $\tau \in (0, \tau^*)$ , or equivalently, such that the spectral radius of the matrix  $\Delta_\tau$  given by (2.5) is smaller than 1 for every  $\tau \in (0, \tau^*)$  (see [10], one of the very few papers addressing this issue). To the best of our knowledge, no satisfactory solution of this problem is available in the literature. Naturally, whilst this problem becomes even more difficult in the presence of plant uncertainty, the adaptive strategy (2.2) “handles” plant uncertainty easily. More precisely, assume that the plant is not exactly known, but that it is known to be contained in a (known) set  $\mathcal{P}$  of plants and that (by using methods from robust control) a feedback  $F$  has been designed which stabilizes all plants in  $\mathcal{P}$  in continuous time (that is, (1.1) is exponentially stable for every system  $(A, B, C)$  in  $\mathcal{P}$ ). Then the conclusions of Theorem 2.2 are valid for every  $(A, B, C)$  in  $\mathcal{P}$ .  $\diamond$

As we have already noted in Example 2.3: in the case of state feedback (that is,  $p = n$  and  $C = I$ ), hypothesis (D) is trivially satisfied (for every sequence  $(\eta_j)_{j \in \mathbb{N}_0}$  and every  $\alpha$ ). In general however, the appearance of hypothesis (D) in statement (ii) of Theorem 2.2 is somewhat unsatisfactory, because it is formulated in discrete-time terms and not in terms of the original continuous-time data. The following definition will be useful in addressing this issue.

**Definition 2.5.** A number  $\delta > 0$  is said to be *pathological* relative to  $A \in \mathbb{R}^{n \times n}$  if, and only if, there exist  $q \in \mathbb{Z} \setminus \{0\}$  and  $\lambda, \mu \in \sigma(A) \cap \{s \in \mathbb{C} : \operatorname{Re} s \geq 0\}$  such that  $\delta(\lambda - \mu) = 2q\pi i$ . Otherwise,  $\delta$  is said to be *non-pathological* relative to  $A$ .  $\diamond$

We shall see that, in Theorem 2.2, hypothesis (D) can be replaced by the following hypothesis.

(D') For every  $l \in \mathbb{N}_0$ ,  $\delta_l$  is non-pathological relative to  $A$ .

**Lemma 2.6.** *If the pair  $(C, A)$  is detectable in continuous time and hypothesis (D') is satisfied, then (D) holds.*

The proof of Lemma 2.6 can be found in Section 4.

The assumption of exponential stability of the continuous-time feedback system (1.1) in Theorem 2.2 trivially implies that  $(C, A)$  is detectable in continuous time. Therefore the following corollary is an immediate consequence of Lemma 2.6.

**Corollary 2.7.** *The conclusions of Theorem 2.2 remain valid if, in the statement of Theorem 2.2, hypothesis (D) is replaced by hypothesis (D').*

The following remark contains some commentary hypotheses (D) and (D').

**Remark 2.8.** (i) The converse of Lemma 2.6 is not correct. Whilst hypothesis (D) implies the continuous-time detectability of  $(C, A)$ , it does, in general, not imply (D'). Consequently, in the context of Theorem 2.2, hypothesis (D) is weaker than hypothesis (D').

(ii) Let  $\alpha$  and  $(\eta_l)_{l \in \mathbb{N}_0}$  be given as in (2.2) and define  $(\delta_l)_{l \in \mathbb{N}_0}$  by (2.4). Then it can be shown that the set

$$\{A \in \mathbb{R}^{n \times n} : \delta_l \text{ is non-pathological relative to } A \text{ for every } l \in \mathbb{N}_0\}$$

is open and dense in  $\mathbb{R}^{n \times n}$  (see [5, Appendix A.1]). Consequently, the probability that, for a randomly chosen matrix  $A \in \mathbb{R}^{n \times n}$ , there exists  $l \in \mathbb{N}_0$  such that  $\delta_l$  is pathological relative to  $A$  is zero.

(iii) Let  $A \in \mathbb{R}^{n \times n}$  and  $\alpha \in (0, 1)$  be given and let  $NP(A, \alpha)$  denote the set of all bounded sequences  $(\eta_l)_{l \in \mathbb{N}_0}$  with  $\inf_{l \in \mathbb{N}_0} \eta_l > 0$  and such that  $\delta_l$  (defined in (2.4)) is non-pathological relative to  $A$  for every  $l \in \mathbb{N}_0$  (that is, hypothesis (D') holds). It is easy to show that  $NP(A, \alpha)$  is open and dense (with respect to the  $\ell^\infty$ -norm) in the set of all bounded sequences  $(\eta_l)_{l \in \mathbb{N}_0}$  with  $\inf_{j \in \mathbb{N}_0} \eta_l > 0$ . As a consequence, the probability that a randomly chosen sequence  $(\eta_l)_{l \in \mathbb{N}_0}$  with  $\inf_{l \in \mathbb{N}_0} \eta_l > 0$  is not contained in  $NP(A, \alpha)$  is zero.  $\diamond$

Part (ii) of Remark 2.8 shows that, if  $\alpha$  and  $(\eta_l)_{l \in \mathbb{N}_0}$  are fixed, then, with respect to  $A$ , (D') is generically satisfied. Similarly, if  $\alpha$  and  $A$  are fixed, then part (iii) of Remark 2.8 shows that, with respect to  $(\eta_l)_{l \in \mathbb{N}_0}$ , (D') holds generically. The same comment applies to hypothesis (D), provided that  $(C, A)$  is detectable in continuous time (the latter is trivially satisfied if the continuous-time feedback system (1.1) is exponentially stable). Consequently, assumptions (D) and (D') imposed in Theorem 2.2 and Corollary 2.7, respectively, are not very restrictive.

We illustrate Theorem 2.2 by an example (including a numerical simulation).

**Example 2.9.** Assume that  $A, B, C$  and  $F$  in system (2.1) are given by

$$A = \begin{pmatrix} -a_1 & 1 & a_2 \\ -1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = I, \quad F = -B^T.$$

Then, for all  $(a_1, a_2, a_3) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$ , the matrix  $A$  is dissipative (that is  $\langle Az, z \rangle \leq 0$  for all  $z \in \mathbb{R}^3$ ) and the pair  $(A, B)$  is continuous-time controllable. Consequently, as is well known, the corresponding continuous-time feedback system (1.1) is exponentially stable (for all  $(a_1, a_2, a_3) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$ ). Hypothesis (D) is trivially satisfied (for every sequence  $(\eta_j)_{j \in \mathbb{N}_0}$  and every  $\alpha$ ) and therefore the conclusions of Theorem 2.2 hold (for all  $(a_1, a_2, a_3) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$ ).

Consider  $A$  with specific parameter values given by  $a_1 = 0$ ,  $a_2 = 1/2$  and  $a_3 = 1$ , in which case  $A$  has eigenvalues 0 and  $\pm i 3/2$  and the eigenvalues of  $A - BB^T$  are approximately  $-0.8836$  and  $-0.5582 \pm i 1.3971$ . Moreover, with  $\alpha$ ,  $(\eta_j)$ ,  $x^0$  and  $\sigma_0$  given by

$$\alpha = 0.3, \quad \eta_j = 1 \quad \forall j \in \mathbb{N}_0, \quad x^0 = (1, 2, 1)^T, \quad \sigma_0 = 0,$$

the evolution of the sampled-data system given by (2.1) and (2.2) is illustrated in Figure 2.1.  $\diamond$

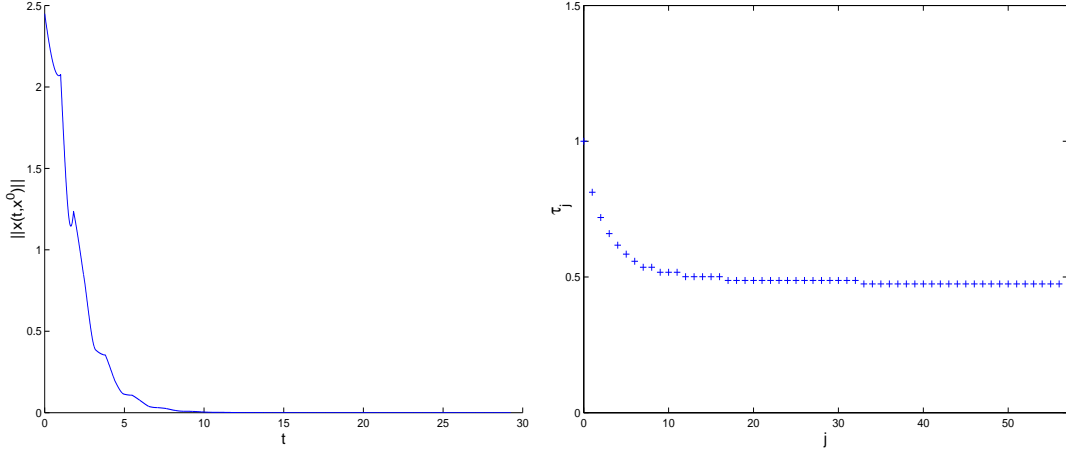


Figure 2.1: Sampled-data control with adaptive sampling period.

### 3 Generalization to dynamic output feedback

Consider a dynamic output feedback system with plant given by

$$\begin{aligned} \dot{x}_p &= A_p x_p + B_p u_p; & x_p(0) &= x_p^0, \\ y_p &= C_p x_p, \end{aligned} \quad (3.1)$$

controller given by

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c u_c; & x_c(0) &= x_c^0, \\ y_c &= C_c x_c + D_c u_c, \end{aligned} \quad (3.2)$$

and feedback interconnection equations

$$u_c = y_p, \quad u_p = y_c, \quad (3.3)$$

where  $A_p \in \mathbb{R}^{n_p \times n_p}$ ,  $B_p \in \mathbb{R}^{n_p \times m}$ ,  $C_p \in \mathbb{R}^{p \times n_p}$ ,  $A_c \in \mathbb{R}^{n_c \times n_c}$ ,  $B_c \in \mathbb{R}^{n_c \times p}$ ,  $C_c \in \mathbb{R}^{m \times n_c}$ ,  $D_c \in \mathbb{R}^{m \times p}$ ,  $x_p^0 \in \mathbb{R}^{n_p}$  and  $x_c^0 \in \mathbb{R}^{n_c}$ . Defining

$$A := \text{diag}(A_p, A_c), \quad B := \text{diag}(B_p, B_c), \quad C := \begin{pmatrix} C_p & 0 \\ D_c C_p & C_c \end{pmatrix}, \quad F := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad (3.4)$$

a routine calculation shows that the continuous-time dynamic feedback system given by (3.1)–(3.3) can be written as

$$\dot{x} = (A + BFC)x; \quad x(0) = x^0 = \begin{pmatrix} x_p^0 \\ x_c^0 \end{pmatrix}, \quad \text{where } x := \begin{pmatrix} x_p \\ x_c \end{pmatrix}. \quad (3.5)$$

Let  $(t_j)_{j \in \mathbb{N}_0}$  be the sampling points to be determined adaptively. As before, we define the associated sampling periods  $\tau_j := t_{j+1} - t_j$  for  $j \in \mathbb{N}_0$ . Consider the corresponding sample-hold discretization of (3.2)

$$\begin{aligned} x_c^d(j+1) &= e^{A_c \tau_j} x_c^d(j) + \int_0^{\tau_j} e^{A_c s} ds B_c u_c^d(j); & x_c^d(0) &= x_c^0 \in \mathbb{R}^{n_c}, \\ y_c^d(j) &= C_c x_c^d(j) + D_c u_c^d(j), \end{aligned} \quad (3.6)$$



together with the feedback interconnection equations

$$u_c^d(j) = y_p(t_j), \quad u_p(t_j + \theta) = y_c^d(j), \quad \forall \theta \in [0, \tau_j), \quad \forall j \in \mathbb{N}_0. \quad (3.7)$$

The adaptive strategy for determining the sampling points is very similar to that in the case of static feedback, the only difference being in the equation for  $(\sigma_j)_{j \in \mathbb{N}_0}$ :

$$\left. \begin{aligned} &\text{for given } \alpha \in (0, 1) \text{ and } (\eta_j)_{j \in \mathbb{N}_0} \in \ell^\infty(\mathbb{N}_0, \mathbb{R}) \text{ with } \inf_{j \in \mathbb{N}_0} \eta_j > 0, \\ &\quad \text{set } t_0 = 0, \quad \text{let } \sigma_0 \geq 0, \\ &\text{and, for } j = 0, 1, 2, \dots, \quad \text{set} \\ &\quad k_j = \lfloor \sigma_j \rfloor, \\ &\quad \tau_j = \max \left\{ \eta_j / (j + 1)^\alpha, \eta_{k_j} / (k_j + 1)^\alpha \right\}, \\ &\quad t_{j+1} = t_j + \tau_j, \\ &\quad \sigma_{j+1} = \sigma_j + \left\| (y_p(t_j), y_c^d(j)) \right\|. \end{aligned} \right\} \quad (3.8)$$

**Remark 3.1.** Remark 2.1 remains true in the context of the adaptive strategy (3.8), provided that, in (2.3),  $\|y(t_j)\|$  is replaced by  $\|(y_p(t_j), y_c^d(j))\|$  and, in item (d) of part (ii),  $(y(t_j))_{j \in \mathbb{N}_0}$  and  $\ell^1(\mathbb{N}_0, \mathbb{R}^p)$  are replaced by  $(y_p(t_j), y_c^d(j))_{j \in \mathbb{N}_0}$  and  $\ell^1(\mathbb{N}_0, \mathbb{R}^{p+m})$ , respectively.  $\diamond$

The sampled-data feedback system given by (3.1), (3.6), (3.7) and (3.8) has a unique solution which will be denoted by

$$\begin{pmatrix} x_p(t_j + \theta; x^0) \\ x_c^d(j; x^0) \end{pmatrix}, \quad \forall \theta \in [0, \tau_j), \quad \forall j \in \mathbb{N}_0. \quad (3.9)$$

The corollary below is the main result of this section. The proof can be found in Section 4.

**Corollary 3.2.** *Assume that the continuous-time dynamic feedback system given by (3.1)–(3.3) (or, equivalently, system (3.5)) is exponentially stable. Then, for every initial state  $x^0 \in \mathbb{R}^{n_p+n_c}$ , the sampled-data feedback system given by (3.1), (3.6), (3.7) and (3.8) has the following properties:*

- (i) *the sequence  $(\tau_j)_{j \in \mathbb{N}_0}$  is ultimately constant, that is, the adaptation of the sampling period terminates in finite time;*
- (ii) *if, additionally,  $\eta_l / (l + 1)^\alpha$  is non-pathological relative to  $A = \text{diag}(A_p, A_c)$  for every  $l \in \mathbb{N}_0$ , then  $\lim_{t \rightarrow \infty} x_p(t; x^0) = 0$ ,  $x_p(\cdot; x^0) \in L^1(\mathbb{R}_+, \mathbb{R}^{n_p})$ ,  $(x_p(t_j; x^0))_{j \in \mathbb{N}_0} \in \ell^1(\mathbb{N}_0, \mathbb{R}^{n_p})$  and  $(x_c^d(j; x^0))_{j \in \mathbb{N}_0} \in \ell^1(\mathbb{N}_0, \mathbb{R}^{n_c})$ .*

## 4 Proofs

To facilitate the proofs of the results in Sections 2 and 3, it is convenient to first state and prove a technical lemma. To this end, consider the sampled-data feedback system (2.1) with a prespecified sequence  $\mathbf{t} := (t_j)_{j \in \mathbb{N}_0}$  of sampling points satisfying

$$t_0 = 0, \quad t_{j+1} > t_j \quad \forall j \in \mathbb{N}_0, \quad t_j \rightarrow \infty \quad \text{as } j \rightarrow \infty.$$

Let  $x(\cdot; x^0, \mathbf{t})$  denote the corresponding solution of system (2.1).

The following lemma shows that if the continuous-time system (1.1) is exponentially stable and if the sampling periods  $\tau_j := t_{j+1} - t_j$  converge to 0 as  $j \rightarrow \infty$ , with rate of convergence sufficiently small, then the sequence  $(x(t_j; x^0, \mathbf{t}))_{j \in \mathbb{N}_0}$  is summable. Here “sufficiently small” means that there exist constants  $M > 0$  and  $\alpha \in (0, 1)$  such that  $\tau_j > Mj^{-\alpha}$  for all  $j \in \mathbb{N}$ .

**Lemma 4.1.** *Assume that the continuous-time feedback system (1.1) is exponentially stable. Let the sequence  $\mathbf{t} = (t_j)_{j \in \mathbb{N}_0}$  be such that  $t_0 = 0$  and  $t_{j+1} > t_j$  for all  $j \in \mathbb{N}_0$ . Set  $\tau_j := t_{j+1} - t_j$  and assume that*

$$\lim_{j \rightarrow \infty} \tau_j = 0 \quad \text{and} \quad \inf_{j \in \mathbb{N}} \tau_j j^\alpha > 0 \quad \text{for some } \alpha \in (0, 1). \quad (4.1)$$

*Then, for every  $x^0 \in \mathbb{R}^n$ , the sequence  $(x(t_j; x^0, \mathbf{t}))_{j \in \mathbb{N}_0}$  is in  $\ell^1(\mathbb{N}_0, \mathbb{R}^n)$ .*

**Proof.** The variation-of-parameters formula yields

$$x(t_{j+1}; x^0, \mathbf{t}) = \left( e^{A\tau_j} + \int_0^{\tau_j} e^{As} ds BFC \right) x(t_j; x^0, \mathbf{t}), \quad \forall j \in \mathbb{N}_0. \quad (4.2)$$

Writing

$$\Delta_j := e^{A\tau_j} + \int_0^{\tau_j} e^{As} ds BFC \quad \text{and} \quad x_j := x(t_j; x^0, \mathbf{t}); \quad \forall j \in \mathbb{N}_0,$$

(4.2) becomes

$$x_{j+1} = \Delta_j x_j, \quad \forall j \in \mathbb{N}_0; \quad x_0 = x^0. \quad (4.3)$$

It follows from the exponential stability of (1.1) that there exists a unique matrix  $P = P^T > 0$ , such that

$$(A + BFC)^T P + P(A + BFC) = -I \quad (4.4)$$

(see, for example, [9, Theorem 18, p.231]). Let  $\|\cdot\|_P$  be the norm on  $\mathbb{R}^n$  defined by

$$\|z\|_P^2 := \langle z, Pz \rangle, \quad \forall z \in \mathbb{R}^n.$$

Using the power series expansion of  $e^{At}$ , we may decompose

$$\Delta_j = I + \tau_j(A + BFC) + \tau_j^2 \Gamma(\tau_j), \quad \forall j \in \mathbb{N}_0, \quad (4.5)$$

where

$$\Gamma(\tau) := \sum_{l=0}^{\infty} \frac{\tau^l}{(l+2)!} A^{l+1} (A + BFC), \quad \forall \tau \geq 0.$$

The boundedness of  $(\tau_j)_{j \in \mathbb{N}_0}$  implies the boundedness of the sequence  $(\Gamma(\tau_j))_{j \in \mathbb{N}_0}$  and hence, invoking (4.3) and (4.5), we conclude that there exists a constant  $L \geq 0$  such that

$$\begin{aligned} \|x_{j+1}\|_P^2 - \|x_j\|_P^2 &= \langle \Delta_j x_j, P \Delta_j x_j \rangle - \langle x_j, P x_j \rangle \\ &\leq \tau_j \langle x_j, [(A + BFC)^T P + P(A + BFC)] x_j \rangle + L \tau_j^2 \|x_j\|^2, \quad \forall j \in \mathbb{N}_0. \end{aligned}$$

Combining this with (4.4) shows that

$$\|x_{j+1}\|_P^2 - \|x_j\|_P^2 \leq (-\tau_j + L\tau_j^2)\|x_j\|^2, \quad \forall j \in \mathbb{N}_0,$$

and therefore, in view of  $\lim_{j \rightarrow \infty} \tau_j = 0$ , we obtain that there exists  $N \in \mathbb{N}$  such that

$$\|x_{j+1}\|_P^2 - \|x_j\|_P^2 \leq -\frac{\tau_j}{2}\|x_j\|^2, \quad \forall j \geq N.$$

Consequently,

$$\|x_{j+1}\|_P^2 \leq \|x_j\|_P^2 - \frac{\tau_j}{2}\|x_j\|^2 \leq \left(1 - \frac{\tau_j}{2\|P\|}\right) \|x_j\|_P^2, \quad \forall j \geq N, \quad (4.6)$$

and hence,

$$\|x_j\|_P^2 \leq \left[ \prod_{l=N}^{j-1} \left(1 - \frac{\tau_l}{2\|P\|}\right) \right] \|x_N\|_P^2, \quad \forall j \geq N+1. \quad (4.7)$$

If  $x_{j_0} = 0$  for some  $j_0 \geq N$ , then it follows from (4.6) that  $x_j = 0$  for all  $j \geq j_0$ , and thus  $(x_j)_{j \in \mathbb{N}_0} \in \ell^1(\mathbb{N}_0, \mathbb{R}^n)$ . Assume now that  $x_j \neq 0$  for all  $j \geq N$ . Then, by (4.6),  $1 - \tau_j/(2\|P\|) > 0$  for all  $j \geq N$ . Moreover, since (4.1) yields  $M := \inf_{j \in \mathbb{N}} \{\tau_j j^\alpha\} > 0$ , we have that  $\tau_j \geq M/j^\alpha$  for all  $j \in \mathbb{N}$ , and thus

$$0 < 1 - \frac{\tau_j}{2\|P\|} \leq 1 - \frac{M}{2\|P\|j^\alpha}, \quad \forall j \geq N.$$

Combining this with (4.7) yields

$$\|x_j\|_P \leq \left[ \prod_{l=N}^{j-1} \left(1 - \frac{M}{2\|P\|l^\alpha}\right)^{1/2} \right] \|x_N\|_P, \quad \forall j \geq N+1. \quad (4.8)$$

Define a positive sequence  $(v_j)_{j \in \mathbb{N}_0}$  by

$$v_j := \prod_{l=N}^{N+j} \left(1 - \frac{M}{2\|P\|l^\alpha}\right)^{1/2} = \prod_{l=N}^{N+j} \left(1 - \frac{\gamma}{l^\alpha}\right)^{1/2},$$

where  $\gamma := M/(2\|P\|)$ . By (4.8), to show that  $(x_j)_{j \in \mathbb{N}_0} \in \ell^1(\mathbb{N}_0, \mathbb{R}^n)$ , it suffices to prove that  $(v_j)_{j \in \mathbb{N}_0} \in \ell^1(\mathbb{N}_0, \mathbb{R})$ . Invoking the inequality  $1 - t \leq e^{-t}$  (which holds for all  $t \in \mathbb{R}$ ), we have

$$\sum_{j=0}^k v_j \leq \sum_{j=0}^k \exp\left(-\frac{\gamma}{2} \sum_{l=N}^{N+j} \frac{1}{l^\alpha}\right) \leq \sum_{j=0}^k \exp\left(-\frac{\gamma(j+1)}{2(N+j)^\alpha}\right), \quad \forall k \in \mathbb{N}_0. \quad (4.9)$$

Since, by (4.1),  $\alpha \in (0, 1)$ , it follows that

$$\exp\left(-\frac{\gamma(j+1)}{2(N+j)^\alpha}\right) \leq \frac{1}{j^2}, \quad \text{for all sufficiently large } j.$$

Hence, the right-hand side of (4.9) converges to a finite limit as  $k \rightarrow \infty$ , showing that  $(v_j)_{j \in \mathbb{N}_0} \in \ell^1(\mathbb{N}_0, \mathbb{R})$ .  $\square$

**Proof of Theorem 2.2.** Let  $x^0 \in \mathbb{R}^n$  be fixed, but arbitrary.

To prove statement (i), we adopt a contradiction argument and suppose that the sequence of sampling periods  $(\tau_j)_{j \in \mathbb{N}_0}$  is not ultimately constant. Then, by Remark 2.1,  $\lim_{j \rightarrow \infty} \tau_j = 0$ . Moreover, invoking the definition of  $\tau_j$  in (2.2), we obtain

$$\tau_j j^\alpha \geq \eta_j \left( \frac{j}{j+1} \right)^\alpha, \quad \forall j \in \mathbb{N}.$$

By assumption,  $\inf_{j \in \mathbb{N}_0} \eta_j > 0$ , and thus,

$$\inf_{j \in \mathbb{N}} \tau_j j^\alpha > 0.$$

Therefore, (4.1) is satisfied and Lemma 4.1 yields that  $(x(t_j; x^0))_{j \in \mathbb{N}_0} \in \ell^1(\mathbb{N}_0, \mathbb{R}^n)$ , and hence,  $(y(t_j))_{j \in \mathbb{N}_0} \in \ell^1(\mathbb{N}_0, \mathbb{R}^p)$ . Invoking again Remark 2.1 shows that  $(\tau_j)_{j \in \mathbb{N}_0}$  is ultimately constant, contradicting the supposition that  $(\tau_j)_{j \in \mathbb{N}_0}$  is not ultimately constant.

To prove statement (ii), we first note that, by the variation-of-parameter formula,

$$x(t_j + \theta; x^0) = \left( e^{A\theta} + \int_0^\theta e^{As} ds BFC \right) x(t_j; x^0), \quad \forall \theta \in [0, \tau_j], \quad \forall j \in \mathbb{N}_0. \quad (4.10)$$

By statement (i), there exists  $N \in \mathbb{N}_0$  such that

$$\tau_j = \tau_N =: \tau, \quad \forall j \geq N.$$

Hypothesis (D) guarantees that the pair  $(C, e^{A\tau})$  is discrete-time detectable. Hence there exists  $H \in \mathbb{R}^{n \times p}$  such that  $e^{A\tau} + HC$  is power stable, i.e., all eigenvalues of  $e^{A\tau} + HC$  are in the open unit disc  $\{s \in \mathbb{C} : |s| < 1\}$ . Setting  $B_\tau := \int_0^\tau e^{As} ds B$ , it follows from (4.10) with  $\theta = \tau$  that

$$\begin{aligned} x(t_{j+1}; x^0) &= e^{A\tau} x(t_j; x^0) + B_\tau FC x(t_j; x^0) \\ &= (e^{A\tau} + HC) x(t_j; x^0) + (B_\tau F - H) y(t_j), \quad \forall j \geq N. \end{aligned}$$

Combining this with the power stability of  $e^{A\tau} + HC$  and the fact that  $(y(t_j))_{j \in \mathbb{N}_0} \in \ell^1(\mathbb{N}_0, \mathbb{R}^p)$  (guaranteed by Remark 2.1), we conclude that  $(x(t_j; x^0))_{j \in \mathbb{N}_0} \in \ell^1(\mathbb{N}_0, \mathbb{R}^n)$ . This implies in particular that

$$\lim_{j \rightarrow \infty} x(t_j; x^0) = 0. \quad (4.11)$$

Setting

$$\bar{\tau} := \sup_{j \in \mathbb{N}_0} \tau_j < \infty \quad \text{and} \quad M := \sup_{\theta \in [0, \bar{\tau}]} \left\| e^{A\theta} + \int_0^\theta e^{As} ds BFC \right\|,$$

we obtain from (4.10) that

$$\|x(t_j + \theta; x^0)\| \leq M \|x(t_j; x^0)\|, \quad \forall \theta \in [0, \tau_j], \quad \forall j \in \mathbb{N}_0.$$

Consequently, by (4.11),

$$\lim_{t \rightarrow \infty} x(t; x^0) = 0.$$

Finally,

$$\int_0^\infty \|x(t)\| dt = \sum_{j=0}^\infty \int_{t_j}^{t_{j+1}} \|x(t; x^0)\| dt \leq M\bar{\tau} \sum_{j=0}^\infty \|x(t_j; x^0)\| < \infty,$$

showing that  $x \in L^1(\mathbb{R}_+, \mathbb{R}^n)$  and completing the proof of statement (ii).  $\square$

**Proof of Lemma 2.6.** By assumption,  $(C, A)$  is continuous-time detectable and  $\delta_l$  is non-pathological relative to  $A$  for all  $l \in \mathbb{N}_0$ . Therefore, by a standard result (see [3, Lemma 8]), the pair  $(C, e^{A\delta_l})$  is discrete-time detectable for all  $l \in \mathbb{N}_0$ , showing that hypothesis (D) holds.  $\square$

**Proof of Corollary 3.2.** Let  $x^0 \in \mathbb{R}^{n_p+n_c}$  be fixed, but arbitrary. Moreover, let the matrices  $B$ ,  $C$  and  $F$  be defined as in (3.4). Invoking the variation-of-parameters formula, we conclude that

$$\begin{pmatrix} x_p(t_j + \theta; x^0) \\ x_c^d(j+1; x^0) \end{pmatrix} = \left[ \begin{pmatrix} e^{A_p\theta} & 0 \\ 0 & e^{A_c\tau_j} \end{pmatrix} + \begin{pmatrix} \int_0^\theta e^{A_p s} ds & 0 \\ 0 & \int_0^{\tau_j} e^{A_c s} ds \end{pmatrix} BFC \right] \begin{pmatrix} x_p(t_j; x^0) \\ x_c^d(j; x^0) \end{pmatrix},$$

$$\forall \theta \in [0, \tau_j), \forall j \in \mathbb{N}_0. \quad (4.12)$$

Since, by continuity of  $x_p(\cdot; x^0)$ ,  $x_p(t_j + \theta; x^0) \rightarrow x_p(t_{j+1}; x^0)$  as  $\theta \uparrow \tau_j$ , we obtain from (4.12), as  $\theta \uparrow \tau_j$ ,

$$\begin{pmatrix} x_p(t_{j+1}; x^0) \\ x_c^d(j+1; x^0) \end{pmatrix} = \Delta_j \begin{pmatrix} x_p(t_j; x^0) \\ x_c^d(j; x^0) \end{pmatrix}, \quad \forall j \in \mathbb{N}_0; \quad \begin{pmatrix} x_p(0; x^0) \\ x_c^d(0; x^0) \end{pmatrix} = x^0, \quad (4.13)$$

where  $\Delta_j := e^{A\tau_j} + \int_0^{\tau_j} e^{As} ds BFC$  with  $A$ ,  $B$ ,  $C$  and  $F$  given by (3.4). Now consider the adaptive sampled-data system defined by (2.1) and (2.2), where again  $A$ ,  $B$ ,  $C$  and  $F$  are given by (3.4) and, furthermore,  $n = n_p + n_c$ . Denoting its solution by  $x(\cdot; x^0)$ , it follows that

$$x(t_{j+1}; x^0) = \Delta_j x(t_j; x^0), \quad \forall j \in \mathbb{N}_0; \quad x(0; x^0) = x^0.$$

Combining this with (4.13) shows that

$$x(t_j; x^0) = \begin{pmatrix} x_p(t_j; x^0) \\ x_c^d(j; x^0) \end{pmatrix}, \quad \forall j \in \mathbb{N}_0.$$

An application of Corollary 2.7 to the sampled-data system defined by (2.1) and (2.2), with  $A$ ,  $B$ ,  $C$  and  $F$  given by (3.4), then shows that  $(\tau_j)_{j \in \mathbb{N}_0}$  is ultimately constant and the sequence  $(x(t_j; x^0))_{j \in \mathbb{N}_0}$  is in  $\ell^1(\mathbb{N}_0, \mathbb{R}^n)$ . In particular,

$$\lim_{j \rightarrow \infty} x(t_j; x^0) = \lim_{j \rightarrow \infty} \begin{pmatrix} x_p(t_j; x^0) \\ x_c^d(j; x^0) \end{pmatrix} = 0. \quad (4.14)$$

Finally, we note that by using (4.12) and (4.14) in combination with an argument similar to that adopted at the end of the proof of Theorem 2.2 (after equation (4.11)), it follows that  $\lim_{t \rightarrow \infty} x_p(t; x^0) = 0$  and  $x_p(\cdot; x^0) \in L^1(\mathbb{R}_+, \mathbb{R}^{n_p})$ , completing the proof.  $\square$

## 5 Conclusions

We have proved that if the controlled continuous-time system  $\dot{x} = Ax + Bu$  with output  $y = Cx$  is exponentially stabilized by the static output feedback  $u = Fy$  and if hypothesis (D) or hypothesis (D') holds, then the corresponding indirect sampled-data control together with the adaptive strategy (2.2) leads to a stable sampled-data system in the sense that, for all initial states, the adaptation of the sampling period terminates after finitely many time steps and the state is integrable and converges to zero as time goes to infinity. Furthermore, we have shown how this result can be generalized to dynamic output feedback.

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